DISSERTATIO ACADEMICA,

LINEAM OMNES DATI GENERIS DETERMINATO SUB ANGULO SECANTEM IN SPATIO TRIUM DIMENSIONUM INVESTIGANS;

QUAM,

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§. I.

In evolvendis solutionibus frequenter obvenientibus quæstionum duarum Mathematicos inter satis celebrium, problematis inquam notissimi *Trajectoriarum*, a primis fere quibus excoli cœpit Mathesis sublimior temporibus agitati, nec non quæstionis *Loxodromiarum*, in arte navigationis præcipuum habentis usum, ejusdem omnino casus generalis non nisi specialem haberi harum utramque applicationem fugere nos non potuit, hæcque igitur duo problemata, quantumvis diversis ex considerationibus vulgo tractata, eodem tamen revera solvenda esse principio universali haut nobis dubium visum est. Cujus utique ambarum quæstionum inter se nexus cum mentionem nullibi factam deprehenderimus, talis autem problematum particularium ad classes generales reductio sua non carere utilitate videatur, ingratum non fore theoretica-

rum

rum hujusmodi disquisitionum amantibus speravimus, si resolutionem quæstionis universalis utramque nuper memoratarum corollarii instar particularis complectentis, paucis proponeremus: quam igitur sequentibus pagellis L. B. offerimus, in rei tamen adeo facilis enodatione ad nullam plane nos adspirare laudem ingenue fatentes.

§. II.

Sint igitur

$$\left.\begin{aligned} u &= o \\ u' &= o \end{aligned}\right\} \quad . \quad . \quad (1)$$

æquationes lineæ generis dati, cui eodem sub angulo occurrere semper ponitur linea quæsita, exsistentibus igitur u, u' functionibus datis coordinatarum rectangularum x, y, z, nec non constantis cujusdam p, parametri nomine designandæ, cujus variatione e mutationibus τῶν x, y, z non pendente infinitas prodire patet lineas particulares ejusdem tamen omnes generis datoque sub angulo a quæsita secandas.

Ponantur ulterius

$$\left.\begin{aligned} v &= o \\ v' &= o \end{aligned}\right\} \quad . \quad . \quad (2)$$

æquationes lineæ ipsius quæsitæ coordinatas inter x, y, z, ad memoratum nuperrime systema pertinennentes, nec non constantem quamdam arbitrariam k, ex situ pendentem puncti cujusdam initialis, quod, determinatum quo fiat problema nostrum, transire ponenda est ipsa quaesita.

Quod si jam brevitatis ergo fiat

$$\frac{du}{dz}\frac{du'}{dy} - \frac{du}{dy}\frac{du'}{dz} = a$$

$$\frac{du}{dx}\frac{du'}{dz} - \frac{du}{dz}\frac{du'}{dx} = b$$

$$\frac{du}{dy}\frac{du'}{dx} - \frac{du}{dx}\frac{du'}{dy} = c$$

$$\frac{dv}{dz}\frac{dv'}{dy} - \frac{dv}{dy}\frac{dv'}{dz} = \alpha$$

$$\frac{dv}{dx}\frac{dv'}{dz} - \frac{dv}{dz}\frac{dv'}{dx} = \beta$$

$$\frac{dv}{dy}\frac{dv'}{dx} - \frac{dv}{dx}\frac{dv'}{dy} = \gamma,$$

tangentes quidem utriusque lineae in punctis quorum coordinatae sunt x, y, z, per aequationes

$$\left.\begin{array}{l} a(y'-y) - b(x'-x) = 0 \\ a(z'-z) - c(x'-x) = 0 \end{array}\right\} \text{ atque}$$

$$\left.\begin{array}{l} \alpha(y'-y) - \beta(x'-x) = 0 \\ \alpha(z'-z) - \gamma(x'-x) = 0 \end{array}\right\}$$

 re-

respective definiri notum est: unde, posito $\theta =$ angulo illi determinato, quem cum quæsita efficere semper ponenda est linea generis dati, prodire manifestum est relationem sequentem

$$(a^2 + b^2 + c^2)(\alpha^2 + \beta^2 + \gamma^2) \operatorname{Cos} \theta^2$$
$$= (a\alpha + b\beta + c\gamma)^2 \quad \ldots\ldots\ldots\ldots \quad (3).$$

Ex ipsa jam indole problematis præsentis conditiones quibus satisfacere debent quæsitæ (2) propius examinando, simplicissimas omnino haberi eas elucebit, summa scilicet earum in eo tantum posita, ut (1), (2), (3) inter se conjunctæ non nisi tres revera diversas constituant æquationes, quantitatum scilicet quinque x, y, z, p, k in iis contentarum duabus indeterminatis semper manentibus: unde sequitur, eliminata ipsa p (1) inter & (3), æquationes ambas resultantes cum (2) conjunctas non nisi duas efficere debere æquationes distinctas inter x, y, z, k, sicque, cum (2) quoque atque

$$\left.\begin{array}{l} \alpha dy - \beta dx = o \\ \alpha dz - \gamma dx = o \end{array}\right\},$$

inter se conjunctæ, duas tantum constituant distinctas, quæsitæ tandem (2) manifesto ejus requiruntur indolis, ut, cum resultatis ex eliminatione ipsius p (1) inter atque

$$(a^2 + b^2 + c^2)(dx^2 + dy^2 + dz^2) \operatorname{Cos} \theta^2$$
$$= (adx + bdy + cdz)^2$$

pro-

prodeuntibus conjunctæ, duas tantummodo inter se diversas æquationes efficiant, unde, ex ipsa definitione integralium ad æquationes differentiales pertinentium, alterum memoratorum resultatorum, quod differentialibus scilicet exemtum, cum integrali alterius completo, constante per integrationem introducta arbitraria vicibus ipsius k fungente, conjunctum, ipsas quæ determinandæ erant (2) perfecte constituere censendum est.

Ex qua quidem problematis nostri generalis analysi, ad claras rei de qua agitur notiones gignendas nobis ut videtur non parum conferente, summam igitur ejus solutionis in eo versari patet, ut, eliminata p tres inter æquationes

$$\left.\begin{array}{l} u = o \\ u' = o \\ (a^2 + b^2 + c^2)(dx^2 + dy^2 + dz^2) \operatorname{Cos} \theta^2 \\ = (adx + bdy + cdz)^2 \end{array}\right\} \ldots (4),$$

instituatur dein integratio harum ultimæ, introducta simul constante arbitraria k: sicque duas prodire perspicitur æquationes ipsas inter x, y, z & k, quæ quæsitæ sunt lineæ omnes in classe (1) contentas dato sub angulo θ secantis.

§. III.

En igitur generalem quæstionis nostræ enodationem ad illustrandum casum unum alterumve spe-

specialem aliquantulum jam tantum accommodandam: quod tamen antequam adgredimur, nonnullas de solutionis præcedentis pro diversis ipsius θ valoribus indole observationes præmittere alienum non erit.

Quod si igitur primum $\theta = 1^q$, fiet $\operatorname{Cos}\theta = o$, hincque (4) in

$$\left.\begin{array}{l} u = o \\ u' = o \\ adx + bdy + cdz = o \end{array}\right\} \quad . \quad . \quad (5)$$

abibunt: quæ simplicissima omnino est forma, allati supra generis calculo ulterius tractanda.

Sin autem $\theta = o$, unde $\operatorname{Cos}\theta = 1$, simplicius multo primo intuitu prodire non videtur systema nostrum (4): attentius vero eo considerato, æquationem tertiam, sub forma

$$(ady - bdx)^2 + (adz - cdx)^2 + (bdz - cdy)^2 = o$$

jam prodeuntem, in sequentes duas sponte dilabi patet

$$\left.\begin{array}{l} ady - bdx = o \\ adz - cdx = o \end{array}\right\},$$

quæ communi hoc in casu gaudere tangente lineam dati generis ipsamque quæsitam evidenter indicant. Hasce igitur ambas cum (1) conjungendo, parame-

trum-

trumque omnes inter quattuor eliminando, tres inter x, y, z æquationes obtinentur, quæ resolutionem casus præsentis complecti censendæ sunt. Heic vero difficultas quædam tironem facile perturbare potest. Integralia scil. completa memoratarum trium ex eliminatione τȣ p resultantium propius examinando, ex nota æquationum differentialium theoria per ipsas datas

$$\left. \begin{array}{l} u = o \\ u' = o \end{array} \right\}$$

ea exhiberi facile perspicitur, parametro p constantis arbitrariæ vicibus fungente. Quonam igitur modo linea quæsita, quæ cum illa generis dati confundenda non est, hoc in casu eruenda? Responsum simplex est. Cum allatæ scilicet nuperrime tres resultantes casus de quo agitur plenam contineant enodationem necesse sit, integralia autem earum completa ad eam non ducere perspexerimus, per *solutiones* earumdem æquationum *particulares* quærendam eamdem jam esse sequitur, quod et inde confirmatur, quod, casu hocce penitius considerato, absque difficultate pateat, datis formis των u & u', per punctum initiale pro arbitrio acceptum duci jam minime posse quæsitam, unde constantem quamdam arbitrariam æquationes solutionem casus præsentis complectentes continere non posse manifestum fit. Solutione igitur partiticulari quæsita solito modo ex integrali completo dato elicita, veram casus $\theta = o$ enodationem ex eliminatione ipsius p quattuor inter

$$\left.\begin{aligned} u &= o \\ u' &= o \\ \frac{du}{dp} &= o \\ \frac{du'}{dp} &= o \end{aligned}\right\} \quad . \; . \; (6),$$

hauriendam esse perspicitur: unde prodire scilicet videmus tres inter x, y, z æquationes nullam involventes constantem arbitrariam, quæ, si *duæ* tantum inter se distinctæ habeantur, curvam hoc in casu quæsitam definient.

Quod si denique neque $= 1^q$, neque $= o$, habeatur angulus datus θ, ob duplex hoc in casu in æquatione tertia systematis (4) differentialium signum duabus quoque solutionibus diversis quæstionem de qua agitur obnoxiam esse apparet: quod ex ipsa ejus indole concludi etiam potuerat, cum perspiciatur scil. absque negotio conditionibus problematis nostri in genere satisfacere debere curvam duorum ramorum, sese in puncto quodam sub angulo 2θ secantium.

§. IV.

§. IV.

Ad usum solutionis nostræ generalis, systemate (4) comprehensæ, ostendendum, fiat ex. gr.

$$u' = z,$$

unde patet lineam generis dati in plano xy totam quantam sitam fore, natura ejus ceterum nihil determinata. Habetur jam

$$a = -\frac{du}{dy}, \quad b = \frac{du}{dx}, \quad c = o,$$

unde (4) prodit

$$\left.\begin{array}{l} u = o \\ z = o \\ \left(\frac{du^2}{dx^2} + \frac{du^2}{dy^2}\right)(dx^2 + dy^2 + dz^2)\,\mathrm{Cos}\,\theta^2 \\ = \left(\frac{du}{dx}\,dy - \frac{du}{dy}\,dx\right)^2 \end{array}\right\},$$

vel simplicius adhuc

$$\left.\begin{array}{l} z = o \\ u = o \\ \frac{du}{dx}\,dx + \frac{du}{dy}\,dy \pm \left(\frac{du}{dx}\,dy - \frac{du}{dy}\,dx\right)\mathrm{Tg}\,\theta = o \end{array}\right\} \quad . \quad . \quad (7),$$

ubi signum superius vel inferius pro lubitu usurpari potest, prout hic vel ille ramorum quæsitæ considerandus est.

Posito $\theta = 1^q$ vel $\theta = o$, fiet

$$\left.\begin{array}{l} z = o \\ u = o \\ \frac{du}{dx} dy - \frac{du}{dy} dx = o \end{array}\right\} \text{vel} \left.\begin{array}{l} z = o \\ u = o \\ \frac{du}{dx} dx + \frac{du}{dy} dy = o \end{array}\right\};$$

adhibita tamen, ut in §. præc. observatum est, hoc in casu æquationis differentialis *solutione* tantum *particulari*.

Quod si v. gr.

$$u = y^m - px^n,$$

allatum nuper (7) fiet

$$\left.\begin{array}{l} z = o \\ y^m - px^n = o \\ my^{m-1} dy - npx^{n-1} dx \mp (npx^{n-1} dy + my^{m-1} dx) \operatorname{Tg} \theta = o \end{array}\right\};$$

hincque, eliminata, uti supra præscriptum est, ipsa p, æquationes quæsitam definientes habentur

$$\left.\begin{array}{l} z = o \\ nydx - mxdy \pm (mxdx + nydy) \operatorname{Tg} \theta = o \end{array}\right\},$$

quarum facile inveniuntur integralia

$$z = o$$

$$L . k \sqrt{(n-m) xy \pm (mx^2 + ny^2) \operatorname{Tg} \theta}$$

$$= \frac{m + n}{\sqrt{4mn \operatorname{Tg} \theta^2 - (n-m)^2}} . \text{Arc.} \operatorname{Tg} \left(\frac{(n-m) x \pm 2ny \operatorname{Tg} \theta}{x \sqrt{4mn \operatorname{Tg} \theta^2 - (n-m)^2}} \right)$$

(si

(si positiva est $4mn \operatorname{Tg} \theta^2 - (n-m)^2$)

$$= \frac{-(m+n)x}{(n-m)x \pm 2ny \operatorname{Tg} \theta}$$

(si $4mn \operatorname{Tg} \theta^2 - (n-m)^2 = o$)

$$= \frac{m+n}{\sqrt{(n-m)^2 - 4mn \operatorname{Tg} \theta^2}} \cdot$$

$$L.\left(\frac{(n-m)x \pm 2ny \operatorname{Tg} \theta - x\sqrt{(n-m)^2 - 4mn \operatorname{Tg} \theta^2}}{\sqrt{(n-m)xy \pm (mx^2 + ny^2) \operatorname{Tg} \theta}}\right)$$

(si negativa est $4mn \operatorname{Tg} \theta^2 - (n-m)^2$),

designante $L.$ Logarithmos Hyperbolicos, nec non k constantem arbitrariam integratione introductam.

Sit e. g. $m = n$; habebuntur

$$\left.\begin{array}{l} z = o \\ \pm \operatorname{Tg} \theta . L . k\sqrt{x^2 + y^2} = \text{Arc. Tg} \dfrac{y}{x} \end{array}\right\},$$

ad Logarithmicam Spiralem pertinentes.

Allata huc usque priorem quæstionum inclytarum in §. I. memoratarum respicere facile patet: ad posteriorem quod attinet, fiat tantum

$$u = y - px$$
$$u' = x^2 + y^2 - \varphi z^2,$$

denotante φ functionem arbitrariam.

 Erit

Erit jam

$$a = 2\varphi z\,.\,\varphi' z,\quad b = 2p\,\varphi z\,.\,\varphi' z,\quad c = 2\,(x + py),$$

posito brevitatis ergo $\frac{d\varphi z}{dz} = \varphi' z.$

Hincque (4) in

$$\left.\begin{array}{l} y - px = o \\ x^2 + y^2 - \varphi z^2 = o \\ (4(1+p^2)\varphi z^2.\varphi' z^2 + 4(x+py)^2)\,(dx^2 + dy^2 + dz^2)\mathrm{Cos}\,\theta^2 \\ = (2\varphi z\,.\,\varphi' z\,dx + 2p\,\varphi z\,.\,\varphi' z\,dy + 2\,(x+py)\,dz)^2 \end{array}\right\}$$

abibunt, quas inter instituta eliminatione ipsius p, factisque reductionibus debitis, ad sequentes satis simplices quæsitam hoc in casu definientes perveniemus

$$\left.\begin{array}{l} x^2 + y^2 - \varphi z^2 = o \\ (dx^2 + dy^2 + dz^2)\,\mathrm{Cos}\,\theta^2 = dz^2\,(1 + \varphi' z^2) \end{array}\right\} \;.\;.\;(8).$$

Quarum utique posteriore, æquationis ope

$$dx^2 + dy^2 = \frac{(xdy - ydx)^2}{x^2 + y^2} + \frac{(xdx + ydy)^2}{x^2 + y^2}$$
$$= \frac{(xdy - ydx)^2}{x^2 + y^2} + \varphi' z^2\,.\,dz^2,$$

ad formam revocata

xdy

$$\frac{xdy - ydx}{x^2 + y^2} = \pm \frac{\sqrt{1 - \text{Cos}\,\theta^2}}{\text{Cos}\,\theta} \cdot \frac{dz}{\varphi z} \cdot \sqrt{1 + \varphi' z^2},$$

easdem (8) sub specie tandem finita

$$\left.\begin{array}{l} x^2 + y^2 - \varphi z^2 = o \\ \text{Arc. Tg}\,\frac{y}{x} = k \pm \text{Tg}\,\theta \cdot \int \frac{dz}{\varphi z} \cdot \sqrt{1 + \varphi' z^2} \end{array}\right\} \ldots (9)$$

exhibere licet, generalem quæstionis de qua agitur resolutionem involvente.

Quod si v. gr. $\varphi z = \sqrt{r^2 - q^2 z^2}$, formam induet (9) sequentem

$$x^2 + y^2 + q^2 z^2 - r^2 = o$$

$$\text{Arc. Tg}\,\frac{y}{x} = k \pm \text{Tg}\,\theta \cdot \int \frac{dz\sqrt{r^2 - (1 - q^2)\,q^2 z^2}}{r^2 - q^2 z^2}$$

$$= k \pm \text{Tg}\,\theta \cdot \left\{ L \cdot \left(\frac{\sqrt{r^2 - q^2 z^2}}{\sqrt{r^2 - (1 - q^2)\,q^2 z^2} - q^2 z} \right) - \frac{\sqrt{1 - q^2}}{q} \cdot \text{Arc. Tg} \left(\frac{\sqrt{r^2 - (1 - q^2)\,q^2 z^2}}{qz \cdot \sqrt{1 - q^2}} \right) \right\}$$

(si positiva est $1 - q^2$)

$$= k \pm \text{Tg}\,\theta \cdot L \cdot \sqrt{\frac{r + z}{r - z}}$$

(si $1 - q^2 = o$)

$= k$

$$= k \pm \operatorname{Tg}\theta . \left\{ L . \left(\frac{\sqrt{r^2 - q^2 z^2}}{\sqrt{r^2 + (q^2 - 1) q^2 z^2} - q^2 z} \right) \right.$$

$$\left. + \frac{\sqrt{q^2 - 1}}{q} . L . \left(\sqrt{r^2 + (q^2 - 1) q^2 z^2} - q z \sqrt{q^2 - 1} \right) \right\}$$

(si negativa est $1 - q^2$).

Quod si in casu ultimo, qui attentione omnium est dignissimus, eliminari ponimus ipsas $\frac{y}{x}$, z & q æquationum ope

$$\frac{y}{x} = \operatorname{Tg} l$$

$$\varphi' z = \frac{-q^2 z}{\sqrt{r^2 - q^2 z^2}} = -\operatorname{Tg}\lambda$$

$$q = \frac{1}{\sqrt{1 - e^2}},$$

duarum nuper allatarum posterior in

$$l = k \pm \operatorname{Tg}\theta . \left\{ L . \operatorname{Tg}(45^\circ + \tfrac{1}{2}\lambda) \right.$$

$$\left. + e . L . \sqrt{\frac{1 - e \operatorname{Sin}\lambda}{1 + e \operatorname{Sin}\lambda}} \right\}$$

$$= k \pm \operatorname{Tg}\theta . \left\{ L . \operatorname{Tg}(45^\circ + \tfrac{1}{2}\lambda) - e^2 \operatorname{Sin}\lambda \right.$$

$$\left. - \tfrac{1}{3} e^4 \operatorname{Sin}\lambda^3 - \tfrac{1}{5} e^6 \operatorname{Sin}\lambda^5 - \&c. \right\}$$

mu-

mutabitur : unde, si l', λ' correspondentes quidam sint arcuum l, λ valores, fiet

$$l'-l=\pm \mathrm{Tg}\,\theta.\left\{L.\left(\frac{\mathrm{Tg}(45^\circ+\frac{1}{2}\lambda')}{\mathrm{Tg}(45^\circ+\frac{1}{2}\lambda)}\right)-(\mathrm{Sin}\,\lambda'-\mathrm{Sin}\,\lambda).e^2\right.$$

$$\left.-\tfrac{1}{3}(\mathrm{Sin}\,\lambda'^3-\mathrm{Sin}\,\lambda^3).e^4-\&c.\right\};$$

seu commodius adhuc, si ipsum l per gradus nonages. exprimere nec non Logarithmos adhibere vulgares placeat,

$$l'-l=\pm\zeta\,\mathrm{Tg}\,\theta.\left\{\varkappa.\mathrm{Log.}\left(\frac{\mathrm{Tg}(45^\circ+\frac{1}{2}\lambda')}{\mathrm{Tg}(45^\circ+\frac{1}{2}\lambda)}\right)\right.$$

$$\left.-(\mathrm{Sin}\,\lambda'-\mathrm{Sin}\,\lambda).e^2-\tfrac{1}{3}(\mathrm{Sin}\,\lambda'^3-\mathrm{Sin}\,\lambda^3).e^4-\&c.\right\}..(10),$$

exsistentibus $\zeta=\dfrac{180}{3{,}14159\,..}$, $\varkappa=L.\ 10=2{,}30258\,..$

Pro Tellure quidem, ubi e^2 parva omnino est fractio, magnitudinis scilicet $\dfrac{1}{152{,}75}$ circiter, terminos seriei (10) e^4 potestatesque superiores involventes semper fere negligi posse, uno alterove instituto calculi experimento probari facile potest.